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The $\lambda\Phi^4$ Miracle:

Lattice data and the zero-point potential

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Abstract:

Recent lattice data for the effective potential of $\lambda\Phi^4$ theory fits the massless one-loop formula with amazing precision. Any corrections are at least 100 times smaller than is reasonable, perturbatively. This is strong evidence for the "exactness conjecture" of Consoli and Stevenson.

1. This note is to draw attention to a very striking result obtained recently from lattice simulations of the four-dimensional $\lambda\Phi^4$ theory [1, 2, 3]. At certain parameter values the shape of the effective potential agrees with the (unrenormalized, massless) one-loop formula to amazing precision: no deviations are found at the 10^{-4} level, *even though perturbatively one would expect corrections of order a few percent.* This result is predicted by the picture of Consoli and Stevenson [4, 5]; from any conventional viewpoint the result is an inexplicable miracle.

My aim here is simply to bring out the main point briefly and dramatically. For a thorough discussion, see Refs. [1, 3]. It will suffice to study just one set of data points (obtained by Cea and Cosmai [2] and published in Table 3 of Ref. [1]). Other data sets amply comfirm the result. My analysis is extremely simple, and the sceptical reader may check it in five minutes with a pocket calculator.

The lattice simulations used the discretized Euclidean action:

$$S_J = \sum_x \left[\frac{1}{2} \sum_{\hat{\mu}} (\Phi_{x+\hat{\mu}} - \Phi_x)^2 + \frac{1}{2} m_B^2 \Phi_x^2 + \frac{1}{4!} \lambda_B \Phi_x^4 - J\Phi_x \right] \tag{1}$$

(where x denotes a generic lattice site and $x + \hat{\mu}$ its nearest neighbours). The data were taken on a 16^4 lattice at $\lambda_B = 6$ and $m_B^2 = -0.45$. (All numerical values are in lattice units: $\lambda_B = 6\lambda_0$, and $m_B^2 = r_0$ in the notation of [1, 3].) The value of m_B^2 was chosen to yield the massless case, in the sense of Coleman and Weinberg [6]. The expectation value, $\phi_B(J) \equiv \langle \Phi \rangle_J$ of the bare field was measured for various input values of the constant source J. See the first two columns of Table 1 below. Viewed in reverse, these data give the value of J at a particular ϕ_B value — and $J(\phi_B)$ is $\mathrm{d}V_{\mathrm{eff}}/\mathrm{d}\phi_B$ [6].

J	ϕ_B	$R(J,\phi_B)$
0.100	$0.506086 \pm (0.91 \times 10^{-4})$	17.562 ± 0.010
0.125	$0.543089 \pm (0.82 \times 10^{-4})$	17.543 ± 0.008
0.150	$0.575594 \pm (0.82 \times 10^{-4})$	17.552 ± 0.008
0.200	$0.630715 \pm (0.62 \times 10^{-4})$	17.549 ± 0.005
0.300	$0.717585 \pm (0.52 \times 10^{-4})$	17.548 ± 0.004
0.400	$0.786503 \pm (0.44 \times 10^{-4})$	17.552 ± 0.003
0.500	$0.844473 \pm (0.41 \times 10^{-4})$	17.552 ± 0.003
0.600	$0.894993 \pm (0.39 \times 10^{-4})$	17.551 ± 0.002
0.700	$0.940074 \pm (0.37 \times 10^{-4})$	17.550 ± 0.002

The lattice data [2,1] and the ratio $R(J,\phi_B) \equiv \phi_B^3 \ln(\phi_B^2/v_B^2)/J$.

[Note that the lattice calculation is unable to access the small J region. The statistical errors increase as J decreases, and finite-volume effects become large [1, 3]. This prevents one from directly accessing the region $\phi_B \approx v_B$ as one would ideally like.]

The "miracle" is the extraordinary precision with which these data fit the simple formula

$$J(\phi_B) = \alpha\phi_B^3 \ln(\phi_B^2/v_B^2) \tag{2}$$

(where α and v_B are constants). To demonstrate this fact one could plot J/ϕ_B^3 versus $\ln \phi_B^2$. An even simpler task, if one uses the value $v_B = 5.783 \times 10^{-4}$ obtained in Ref. [1]'s fit, is to check that the ratio $R(J, \phi_B) \equiv \phi_B^3 \ln(\phi_B^2/v_B^2)/J$ is constant. This is done in the third column of Table 1. The constancy of R is evident. Fig. 1 shows this graphically, on a fine scale. Each data point lies within 1.1σ of a common value, $R = 17.551$. The individual statistical errors are between 0.06% and 0.01%. There is no sign of any systematic deviation from constancy at the 0.01% level.

2. This is the empirical result. It accords beautifully with the picture of Ref. [4, 5], in which the form of the effective potential is given exactly, in the continuum limit, by the "zero-point potential" (ZPP) — the classical potential plus the zero-point energy of free-field fluctuations. In the "massless" (or "classically scale-invariant") case this gives

$$V_{\rm eff} \propto \phi_B^4(\ln(\phi_B^2/v_B^2) - \tfrac{1}{2}). \tag{3}$$

The value of the constant of proportionality here is not physically significant, since ϕ_B has to be renormalized. Differentiating this formula yields (2).

In the general case [5] $V_{\rm eff}$ also has a ϕ_B^2 term, giving an extra linear term in the expression for J. This term is absent in the "massless" case, and Ref. [1] found that, at $\lambda_B = 6$, one needs $m_B^2 = -0.45$ to pick out this special case. This finding is completely consistent with Brahm's analysis [7] of independent lattice data. [Ref. [3] has taken data, at $\lambda_B = 3$, using Brahm's central value $m_B^2 = -0.2279$ to avoid any possible bias when selecting the "massless" case. The agreement with (2) is equally spectacular.]

3. Now let us consider whether the results can be explained by conventional theoretical ideas. The loop expansion (after mass renormalization by normal ordering, but before coupling-constant renormalization) would give [6]

$$V_{\rm eff} = \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda_B^2\phi_B^4}{256\pi^2}\left(\ln\left(\frac{\frac{1}{2}\lambda_B\phi_B^2}{\Lambda^2}\right) - \frac{1}{2}\right) + \ldots, \tag{4}$$

where Λ is the ultraviolet cutoff. Differentiating and dividing by ϕ_B^3 yields, at $\lambda_B = 6$:

$$J/\phi_B^3 = 1 + \frac{9}{16\pi^2}\ln\left(\frac{\frac{1}{2}\lambda_B\phi_B^2}{\Lambda^2}\right) + \ldots. \tag{5}$$

If the "$+\ldots$" (2-loop and higher) terms were negligible, then one could absorb the leading term, 1, into the scale of the logarithm and obtain Eq. (2) with $\alpha^{-1} = 16\pi^2/9 = 17.546$, close to the empirical value 17.551. [One could actually adjust v_B to remove this small discrepancy. Using $v_B = 5.7943 \times 10^{-4}$ (see [1], Table 4) leads to a figure virtually identical to Fig. 1 except for a shift in the vertical scale.]

The problem is that, from a conventional viewpoint, we do *not* expect the "$+\ldots$" terms to be negligible. For the quoted data, J/ϕ_B^3 differs by roughly 20% from its leading-order value, 1. This means that the one-loop correction is about 20%, and since the leading logs recur in a geometric series, one would naturally expect the 2-loop term to be of order $(20\%)^2 = 4\%$. Thus, the ratio $R(J, \phi_B)$ ought to show a systematic deviation from constancy at the level of a few percent. In fact there is no sign of any such deviation at the 0.01% level.

The preceding argument ignored renormalization. Conventional renormalization introduces a renormalized coupling constant $\lambda_R(\mu) = \lambda_B(1 - b_0\lambda_B \ln(\Lambda/\mu) + \ldots)$, where $b_0 = 3/(16\pi^2)$. "RG-improvement" tells us (i) to re-sum the series of leading logs (i.e., $(1 - x + \ldots) \to 1/(1 + x) + \ldots$, where $x = b_0\lambda_B \ln(\Lambda/\mu)$, and (ii) to choose the renormalization scale μ to be of order ϕ (there is no distinction between ϕ_B and ϕ_R at this level). The "RG-improved" result at the one-loop level is then basically the classical result with λ_B replaced by the running coupling constant [6]:

$$\lambda_R(\phi) \equiv \frac{\lambda_{R(M)}}{1 - \frac{3\lambda_R(M)}{32\pi^2} \ln \frac{\phi^2}{M^2}}, \tag{6}$$

where M is some arbitary renormalization scale. ($\lambda_R(M{\approx}\Lambda)$ is λ_B.) This would predict

$$J = \lambda_R(\phi)\phi^3/6. \tag{7}$$

One can re-arrange this to yield

$$\ln\phi + \frac{8\pi^2}{9}\frac{\phi^3}{J} = \text{const.} \tag{8}$$

Computing the left-hand side from the data gives values that steadily decrease from 10.691 ± 0.006 to 10.350 ± 0.001. The deviation from constancy is a few percent — but that is much greater than the errors allow. [Ref. [3] has tried fits to the full 2-loop formula, including an adjustable mass parameter; the χ^2/d.o.f is still very poor, 152/14.] In one sense, there is nothing wrong because one expects corrections of relative order $\lambda_R/(16\pi^2)$; a few percent. However, the point remains that the precise agreement with Eq. (2) is completely baffling from a conventional viewpoint. Why does the "naive one-loop" result

fit the data perfectly at the 0.01% level, while the supposedly "improved" formula works only at the few percent level?

That is the "miracle" and, but for a few remarks, I shall leave the matter there for the reader to reflect upon. There is much more data than I have discussed here, including some for the O(2)-symmetric case. It all supports the same conclusion. I can see no way that such precise agreement could arise by accident, or from any kind of mistake. All that need be said about the latter possibility is that the lattice calculation is well-defined and should be eminently reproducible.

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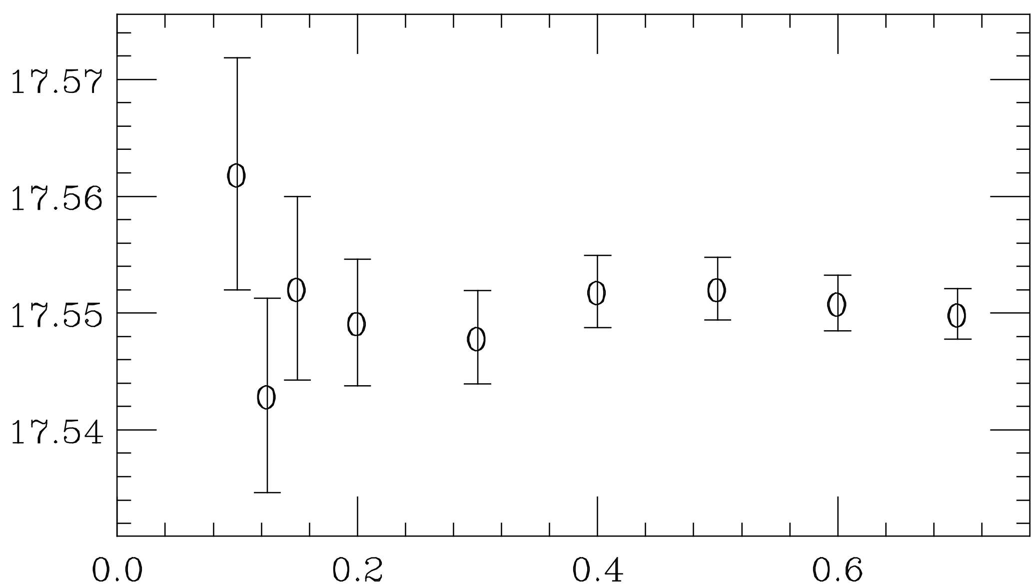
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